Avalanches in Fractional Cascading

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Abstract

This paper studies the distribution of avalanches in fractional cascading, linking the behavior to studies on self-organized criticality, in particular, the power law behavior of the Bak-Tang-Wiesenfeld sandpile model. Unlike the sandpile model, however, we prove that fractional cascading does not exhibit abelian properties. While fractional cascading has maximum gap size as a system parameter that can be used to tune the system behavior, it does not need very fine tuning for the system to display behavior characteristic of self-organized criticality.

1 Introduction

Fractional cascading is an efficient strategy for dealing with iterative searches that achieves optimal time in linear space [3]. Inherent to the technique are characteristics that directly parallel the Bak-Tang-Wiesenfeld sandpile model, which has been shown to produce several characteristic features observed in natural complexity — in particular, power law behavior — in a way that can be linked to critical-point phenomena sometimes described as on the "edge of chaos." In this paper, we provide numerical evidence that the avalanches in fractional cascading exhibit behavior analogous to the sandpile power law behavior in the two- and three-dimensional cases, and characteristic of *self-organized criticality*.

In 1987, Bak, Tang, and Wiesenfeld introduced the concept of self-organized criticality, realizing it with their sandpile model [1]. Self-organized criticality is considered to be one of the mechanisms by which complexity arises in nature, and its concepts have been applied across a diverse array of fields. It describes a system that forms by self-organization to approach a critical point, and then maintains itself at that critical point on the border of stability and chaos (hence the term "edge of chaos"). Such a system is not in balance, where it would be predictable and have states of equilibrium in which stable, small disturbances have only local effect, nor is it in chaos, where it would be entirely unpredictable. These systems are governed by simple rules, but exhibit complex behavior.

The structure that governs fractional cascading shares a lot with systems that display self-organized criticality. There is a threshold (maximum gap size), and pressure builds in the system until it exceeds the threshold. Inserting new elements into a fractional cascading structure naturally progresses it to a state in which small perturbations may have small effects on the system, or may cause system-wide effects (avalanches). However, this behavior in fractional cascading has not been very well studied. Through simulations on fractional cascading structures on path graphs (no spatial structure, so no self-organized criticality), and two- and three-dimensional grids (interesting dynamics with power law correlation functions), we provide numerical evidence linking the behavior of the sandpile model and fractional cascading. In particular, we study the distribution of the size of avalanches in fractional cascading and show that empirical distributions follow power laws when the underlying structure has a relatively high spatial degree of freedom, as in the two- and three-dimensional cases. This is characteristic of a system in a self-organized critical state. The tuning parameter in fractional cascading — gap size — does not need to be finely tuned, but has an admissible range in which it must lie for the fractional cascading system to exhibit self-organized criticality. With such behavior, fractional cascading exhibits major features of self-organized criticality. We also prove that fractional cascading does not have abelian properties, unlike the sandpile model, which makes theoretical analysis considerably more difficult.

In Section 2 of this paper, we discuss the Bak-Tang-Wiesenfeld sandpile model, and in Section 3 we give an overview of fractional cascading, compare it to the sandpile model, and prove that its avalanche behavior is non-abelian. Sections 4 and 5 present our results: Section 4 gives simulation results in the one-dimensional case, and Section 5 simulation results in the two- and three-dimensional cases in which fractional cascading exhibits behavior characteristic of self-organized criticality. We also discuss power law exponents and boundary effects from evolving the system on a finite lattice. A short discussion and possibilities of further directions are given in Section 6.

2 The Bak-Tang-Wiesenfeld Sandpile Model and Self-Organized Criticality

In their 1987 paper, Bak, Tang and Wiesenfeld first coined the term "self-organized criticality," arguing that the dynamics which gives rise to the robust power-law correlations seen in the non-equilibrium steady states in nature evolve naturally into a state at the boundary between stable and unstable states ("the edge of chaos"), and must not involve any fine-tuning of parameters [1]. This is in contrast to critical point behavior at phase transitions in equilibrium statistical mechanics, which can only be reached by fine-tuning of a parameter such as temperature.

Bak et al. presented the sandpile model, a simple example of a driven system displaying self-organized criticality. The sandpile model is a simple cellular automaton characterizing the macroscopic behavior of dry sand. It is defined on a finite lattice — for simplicity, we take this to be a two-dimensional square grid — whose sites correspond to sand piles. At each site of the grid, there is a positive integer value z_i representing the height of the sandpile at that site. The system evolves in discrete time; at each point in time, a site is picked randomly and its height is increased by one (i.e. a grain of sand is added to the sandpile). If its height exceeds a critical height $z_c = 4$, the site has reached an unstable state and collapses, transferring its four grains of sand to each of its four neighboring sites. If there are any unstable sites remaining, they too collapse, until all sites are stable. In case of toppling at a site on the boundary of the grid, the grains that are to fall "outside" the grid are removed from the system. The process converges to a stable configuration in a finite number of time steps on any finite lattice [2].

Thus, for a grid on which the sandpiles are all of height less than z_c, the system is stable: the addition of a small amount of sand will cause only a weak response. In contrast, adding a small amount of sand to a grid in which the average sandpile height is greater than z_c often results in an "avalanche" of size on the order of the size of the grid. However, in a grid in which the average height is z_c, the response to addition of sand is less predictable. Random placement of sand at a particular site may have no effect, or it may cause an avalanche of intermediate size, or an avalanche that affects the entire grid; there is no correlation between the response to a perturbation and the details of a perturbation. Such a state is critical.

This sandpile model is attracted to its critical state, without any fine tuning of a system parameter [1, 2]. This contrasts with earlier examples of critical phenomena, such as the phase transitions between solid and liquid, or liquid and gas, where the critical point can only be reached by precise tuning (of temperature, for phase transitions). Thus in the sandpile model, the criticality is self-organized.

2.1 Self-Organized Criticality in Nature

We use the term *self-organized criticality* in accordance with Dhar in [7], for non-equilibrium steady state of systems with many degrees of freedom, having a steady drive, but where relaxation (e.g. avalanches in sandpiles) occurs in irregular bursts. Self-organized criticality has become established as a strong candidate for explaining a number of natural phenomena. For example, in earthquakes, the build-up of stress due to tectonic motion of the continental plates is a slow steady process, but the release of stress occurs sporadically in bursts of various sizes. With rain, the sun's heat provides a steady drive causing evaporation of water, and the relaxation is the irregular burst-like events of rain. The ideas of self-organized criticality have been applied in a wide variety of systems, including electrical noise, financial market fluctuations, forest fires, solar flares, biological evolution, and epidemics.

3 Fractional Cascading

Fractional cascading is an algorithmic technique for searching several sets at once, which often arises in the solution to query problems, developed by Chazelle and Guibas in 1986 [3, 4]. Consider the problem of identifying a word of unknown origin. One could look up the word in as many dictionaries as it will take to find it, but this is quite a repetitive search. Fractional cascading provides a structure that organizes the dictionaries so that we can find the word by looking it up in one dictionary and then jumping into other dictionaries in constant time. Here, we give an overview of the fractional cascading structure, focusing on the elements that directly contribute to its sandpile-like dynamics.

There are many parallels between grains of sand falling in the sandpile model and inserting new elements (words, in the example of dictionaries) into a fractional cascading data structure, though the description of state in fractional cascading is more complex. The underlying structure of fractional cascading is an undirected, connected simple graph $G = (V, E)$, with maximum degree d and each node $v \in V$ associated with a catalogue C_v which corresponds to a set to be searched. Each catalogue is an ordered collection of records, where each record has an associated key value $\in \mathbb{R}$. Let $n = \sum_{v \in V} |C_v|$ (the size of the catalogue graph).

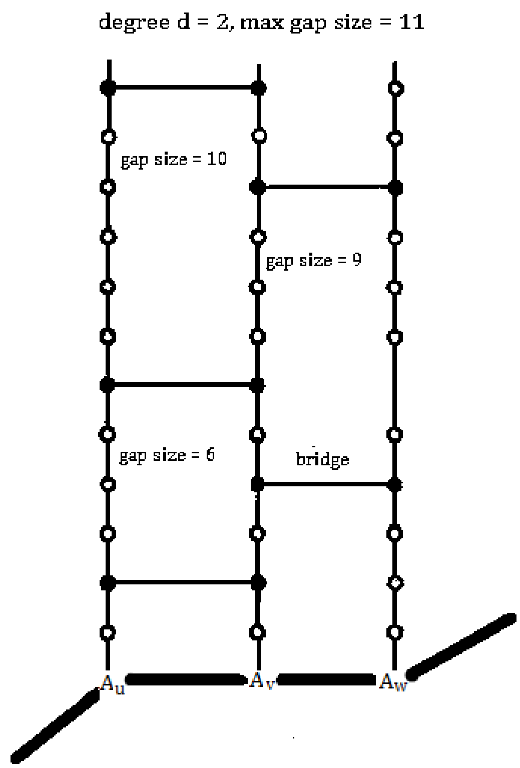
A query to the structure is a pair (x, π), where x is a key value and π a path of G. Instead of performing multiple binary searches at each catalogue along π for x (too much time), or performing a binary search upon a merged master catalogue $M = \bigcup\limits_{v \in V} C_v$ and searching for the appropriate corresponding catalogue which x came from (too much space), the structure is preprocessed so that we only perform one binary search at the beginning, and then at each node v along the search path, locate x in C_v with additional effort that only depends on the maximum degree d of G. Hence with fractional cascading, we can construct data structure in $O(n)$ space and time which allows lookups (x, π), where x is a key value and π a path of length p in G, to be performed in $O(p \log d + \log n)$ time [3].

The catalogues are correlated with bridges and gaps so that a look-up in a catalogue aids to look-up in another. The evolution of bridges and gaps in the structure as new records are added follows rules analogous to the addition of grains of sand to the sandpile model. Each original catalogue C_v is enlarged with additional records to produce an augmented catalogue A_v (also an ordered list of records). Augmented catalogues for neighboring nodes in G contain a number of records with common values; corresponding pairs of records are linked together as *bridges* to correlate locations in the two catalogues. Thus each bridge is associated with a unique edge of G. A pair of consecutive bridges associated with the same edge $e = (u, v)$ defines a *gap*. Let a_u and b_u be two consecutive bridge records in A_u, and a_v and b_v the corresponding bridge records in A_v. The gap defined by these two bridges consists of the records between a_u and b_u (exclusive) and the records between a_v and b_v (exclusive). To ensure constant time traversal between neighboring catalogues, a gap invariant is enforced: no gap can exceed cd in size, where c is a relatively small constant.

Then to search for a value x on a path π, we can perform an initial search on one of the augmented catalogues A_{v_0} in $O(\log n)$ time to find the place of x in A_{v_0}, and then to find the position of x in a neighboring catalogue, A_{v_1}, we travel up A_{v_0} until a bridge to A_{v_1} is encountered ($\leq cd$ steps, due to the gap invariant), cross the bridge to A_{v_1}, and then go downwards on A_{v_1} until the position of x is found (also $\leq cd$ steps, since it must be within the gap). As augmented catalogue records point to successor records (by value) in the original catalogues, we can find the position of x in C_v from the position of x in A_v in one step. This process allows for efficient lookup queries. Further elaboration on searching can be found in [3].

Adding a new record to the structure may violate the gap invariant. Any gaps that

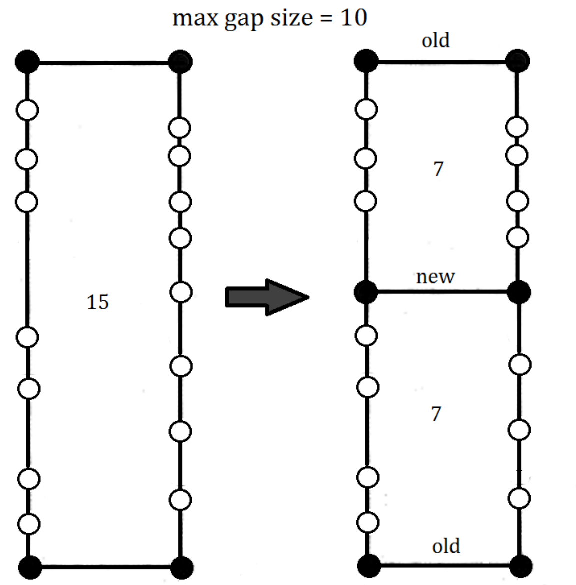
Figure 1: Gaps and bridges [3].



violate the gap invariant must be split into smaller ones. To split a gap associated with edge $e = (u, v)$, consider the ordered list of all the records in the gap, and find the median of the values of the records. Without loss of generality, suppose that the median is on catalogue A_u. Then we make a copy of this record and insert it into A_v, and create a bridge between the median record and its copy. Figure 2 shows the gap-splitting process. This process is repeated for all gaps that violate the gap invariant, until all gap invariants are restored. Further detail about fractional cascading can be found in [3, 4].

Copying median records over to split these gaps may cause additional insertions in neighboring catalogues, producing "avalanches" akin to those in the sandpile model. Where number of sites toppled and distinct number of sites toppled in the sandpile model have been studied and shown to follow power laws, we can look at the number of gaps broken and the

Figure 2: The gap splitting process.



distinct nodes hit in an avalanche (a node v is considered hit if a gap associated with an edge incident with v is split).

3.1 Comparison with the Sandpile Model

The avalanches that occur in irregular bursts are heavily reminiscent of the Bak-Tang-Wiesenfeld sandpile model and self-organized criticality. However, there are a few key differences in the evolution of the fractional cascading structure.

Figure 3: Analogy between fractional cascading and the sandpile model.

	Fractional Cascading	Sandpile
driving force	adding new records	adding sand
relaxing force	gap invariant restoration	gravity
event	gap splitting	sand toppling

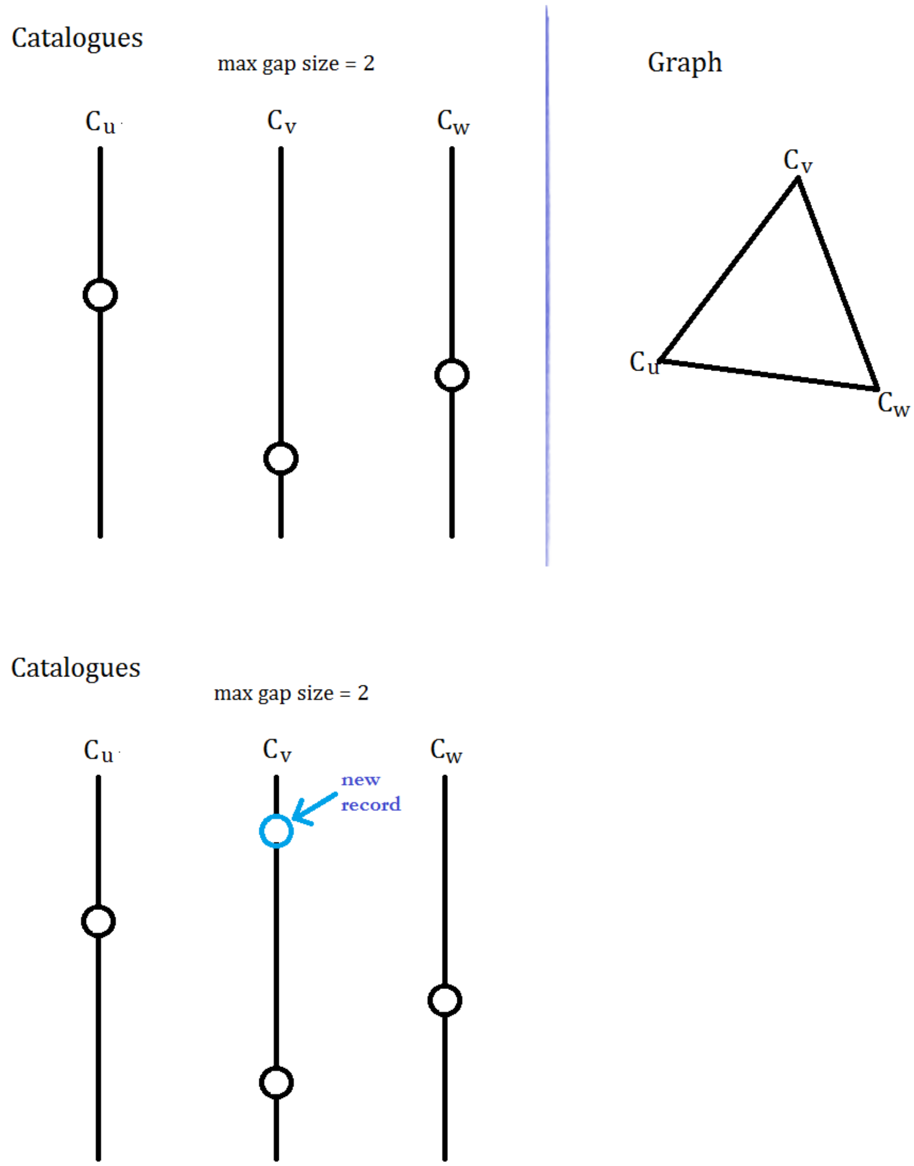
3.1.1 Tuning Parameters

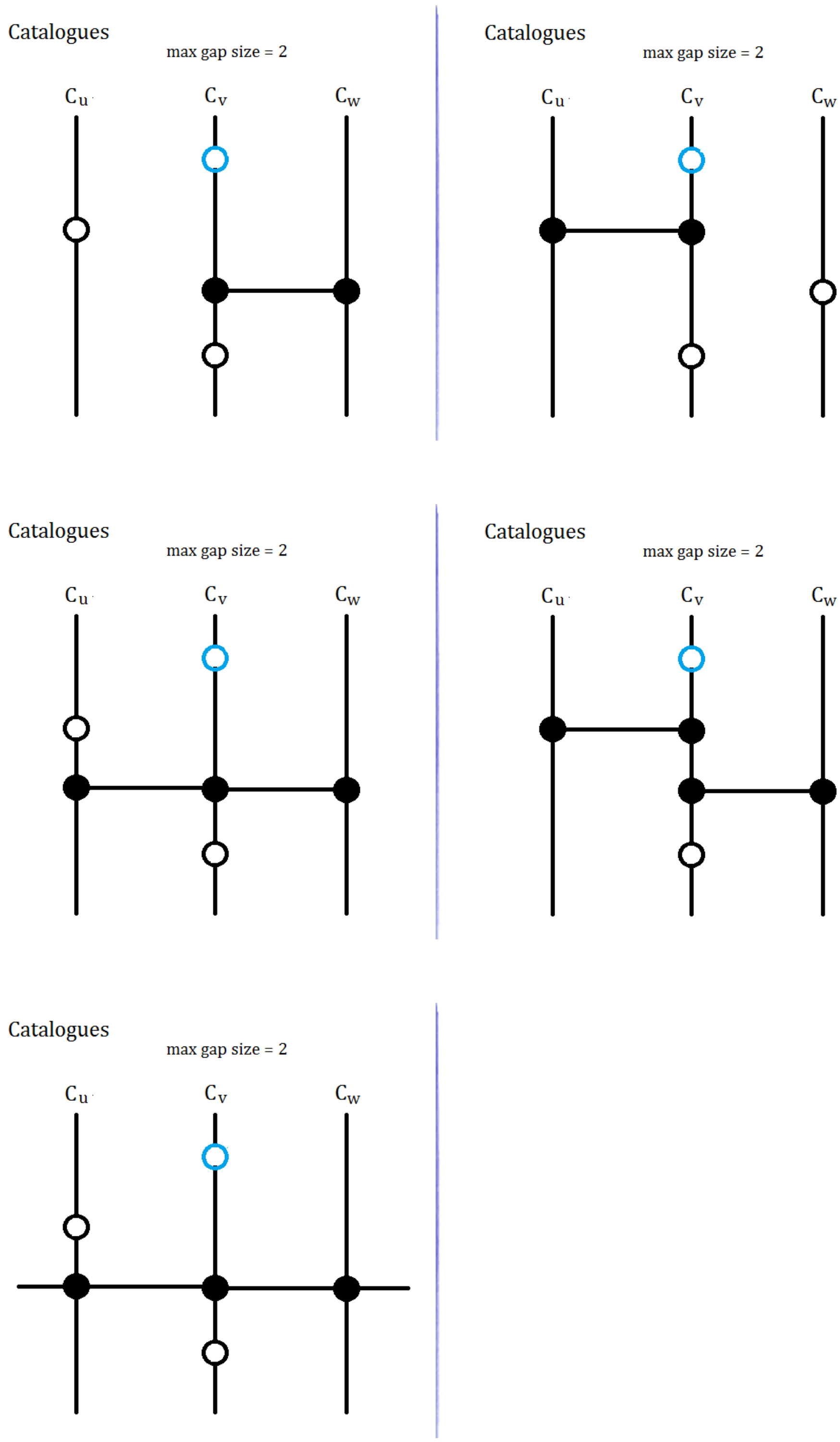
Unlike the sandpile model, which is attracted to its critical state without any fine-tuning of a system parameter, the maximum gap size in fractional cascading lends itself as a parameter that may require some adjustment. An extremely low gap size may result in infinite avalanches if the underlying graph has cycles, since we always insert a new record each time a gap is split. A very high gap size may not result in any avalanches. This kind of behavior seems to parallel that of critical point phenomena that require fine-tuning of parameters, such as the temperature in phase transitions, or the probability a site is open in percolation theory. However, further study in Section 5.3 suggests that instead of requiring gap size to be fine-tuned precisely, there exists a range that the gap size can be and still allow the fractional cascading to exhibit critical point behavior.

3.1.2 Abelian Properties

The sandpile model has also been show to have an important abelian property which simplifies its analysis considerably: if multiple sandpiles have exceeded their critical heights, the order in which they collapse does not affect the end distribution of sand in the system [7]. However, the description of state is much more complex in fractional cascading, as we take into account distinct records and the placement of bridge relationships between catalogues. Fractional cascading does not have an abelian property with regard to the order of processing of gaps that violate the gap invariant, as we prove Figure 4.

Figure 4: Example in which order of processing affects the end state.





Due to the lack of abelian nature, measuring the size of the avalanches by number of gaps broken during the avalanche is slightly less reliable of a parallel to the number of sites toppled during a sandpile avalanche, but it seems unlikely that we run into a significant number of situations in which the order of processing produces drastically different amounts of bridges.

4 The One-Dimensional Case

In this section, we study fractional cascading on a path graph. The one-dimensional path severely limits the amount of freedom an avalanche may have; it can only follow the one path. Because of this limited freedom, we do not expect power law behavior in the one-dimensional case. Rather, we can draw parallels between the size of an avalanche and the length of consecutive carries in binary addition. Consider adding 1 to a binary integer. Consecutive carries occur along a sequence of 1s, and the chance of having a sequence of 1s of length x is 2^{-x}. Thus we conjecture that size of the avalanches follow an exponential distribution.

This is supported by numerical tests. Figures 5 and 6 show an empirical probability mass function for the distinct number of nodes affected in an avalanche and the number of gaps broken during an avalanche, respectively. Ten million records with random values were added randomly to a fractional cascading structure on a path graph with 100 nodes, and maximum gap size $2\Delta(G)$, where $\Delta(G)$ denotes the maximum degree of G. The empirical probabilities were fitted to power law and exponential distributions by minimizing the sum of squared residuals (SSR) as well as chi-squared test statistic, defined as follows, and visually deciding between any conflicts.

For empirical data $\{y_1, ..., y_k\}$ and predicted values $\{f(x_1), ..., f(x_k)\}$, where x_i is the ith value of the explanatory variable, in this case avalanche size, the sum of squared residuals, a measure of the discrepancy between the data and an estimation model, is defined to be

$$SSR = \sum_{i=1}^{k} (y_i - f(x_i))^2 \tag{1}$$

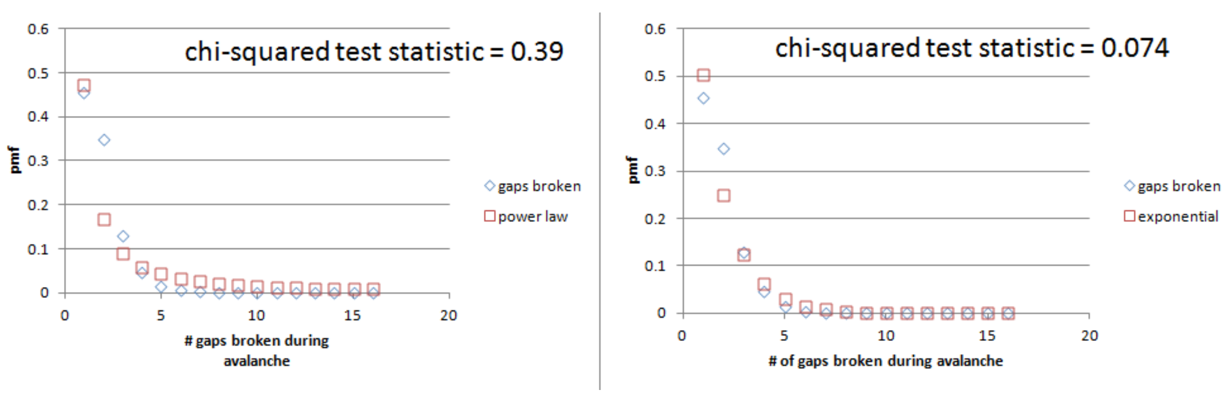
A small SSR indicates a tight fit of the model to the data.

The chi-squared test statistic, indicating goodness of fit (smaller is a better fit), is defined to be

$$\chi^2 = \sum_{i=1}^{k} \frac{(y_i - f(x_i))^2}{f(x_i)} \tag{2}$$

Figure 5: 100 node path graph, 10 million random records: number of distinct nodes hit vs x^{-2} (left) and vs $e^{-.7x}$ (right)

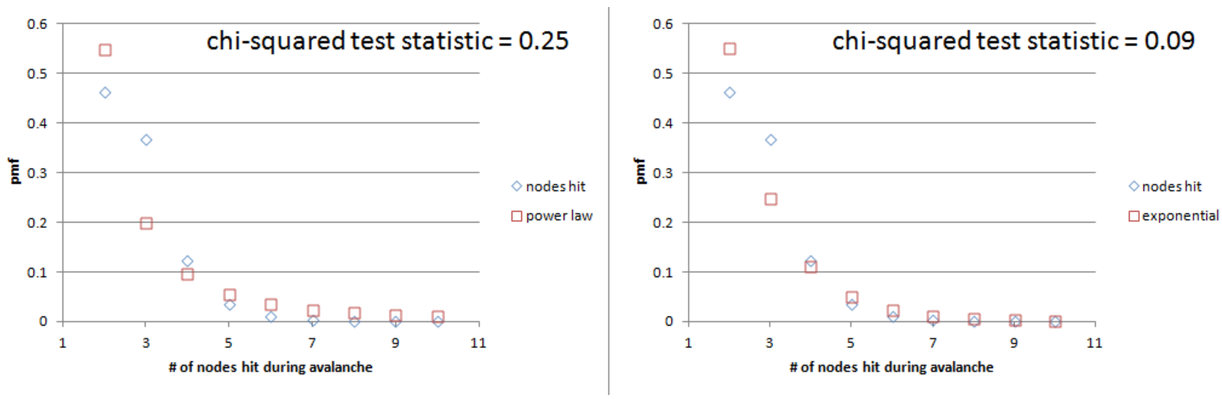
Figure 6: 100 node path graph, 10 million random records: number of gaps broken vs $x^{-1.5}$ (left) and vs $e^{-.7x}$ (right)



For the number of distinct nodes hit: against $e^{-.7x}$, we have $SSR = 0.015$, and $\chi^2 = 0.093$, whereas against x^{-2}, $SSR = 0.033$, and $\chi^2 = 0.33$. For the number of gaps broken: against $e^{-.7x}$, $SSR = 0.013$, and $\chi^2 = 0.074$, whereas against $x^{-1.5}$, $SSR = 0.039$, and $\chi^2 = 0.39$.

From these simulations, the distributions are closer to exponentials, as expected. Varying the number of nodes in the path graph and the number of records inserted produce similar results, as in Figure 7, on a path graph with 10000 nodes. (Due to lack of time, we prioritized the more reliable measure of distinct number of nodes affected, and did not calculate the size of avalanches by number of gaps broken.) A larger number of nodes will reduce any boundary effects from the endpoints with lesser degree. There is no self-organized criticality, which is analogous to the one-dimensional sandpile model; in both, the critical state has no spatial structure, with more predictable correlation functions.

Figure 7: 10,000 node path graph, 100,000 random records: number of distinct nodes hit vs $x^{-2.5}$ (left) and vs $e^{-.8x}$ (right)

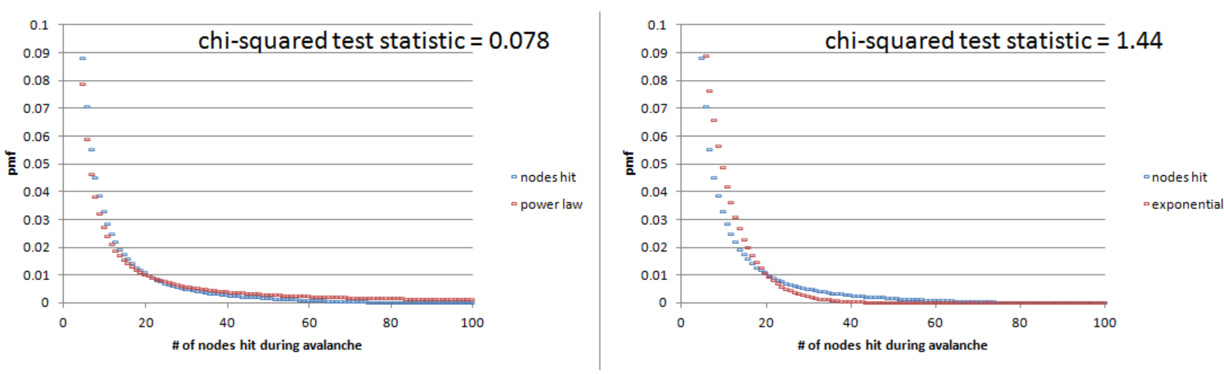


5 Avalanches in Two and Three Dimensions

In two dimensions, we consider catalogues arranged in a square grid. In this case, a single neighboring catalogue is not enough to stop an avalanche: the gaps corresponding to all the neighboring catalogues must satisfy the gap invariant for an avalanche to stop. It is also possible for an avalanche to cycle around in the grid. For these higher dimensions, we have greater spatial degree of freedom, and an avalanche can amplify, spreading from the origin of the avalanche to its neighbors, and then to *their* neighbors in a chain reaction. The system becomes stable when signal cannot be communicated through infinite distances; there will be no length scale and no time scale. That is, a given perturbation can result in anything from an increase in the catalogue size of one catalogue to a large avalanche. Hence we can expect that the system approaches, through a self-organizing process, a critical state with a power-law correlation function.

Simulations on 9×9 and 10×10 grids, with maximum gap size $2\Delta(G)$, by randomly adding in tens of thousands to a million records, produced empirical probability distributions $D_n(x) = \mathbf{Pr}[x$ distinct nodes are affected by an avalanche$]$ that fit very well with the power law $D_n(x) \approx x^{-1.3}$, and $D_g(x) = \mathbf{Pr}[x$ gaps are broken in an avalanche$]$ that fit very well with the power law $D_g(x) \approx x^{-1.2}$. Figures 8 and 9 show these results — $D_n(x)$ and $D_g(x)$, respectively — for a 10×10 grid with 1 million random records and maximum gap size $2\Delta(G)$, against both a power law and an exponential distribution. As before, the empirical probabilities were fitted to power law and exponential distributions by minimizing the SSR (Eq. 1) and chi-squared test statistic (Eq. 2), and visually deciding between any conflicts.

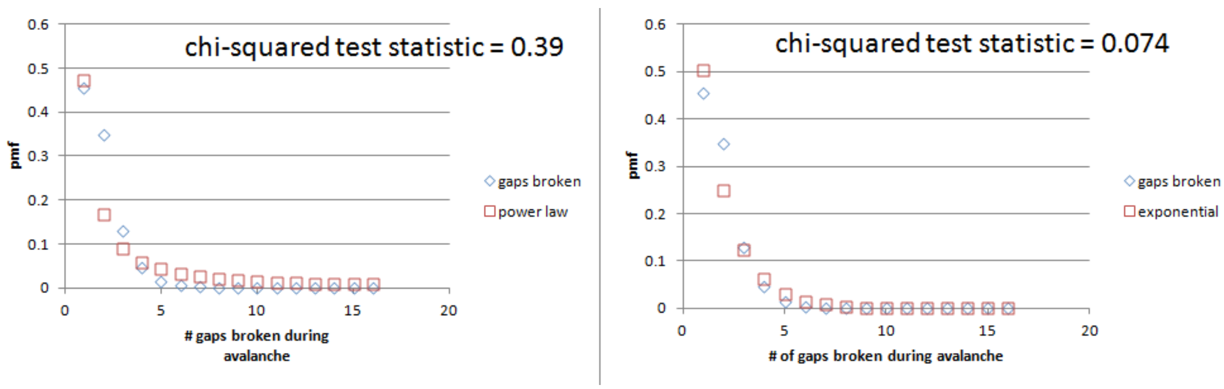
Figure 8: 10×10 grid, 1 million random records: number of distinct nodes hit vs $x^{-1.3}$ (left) and vs $e^{-.15x}$ (right)



For the number of distinct nodes hit: against $x^{-1.3}$, $SSR = 0.001$, and $\chi^2 = 0.078$, whereas against $e^{-.15x}$, $SSR = 0.008$, and $\chi^2 = 1.44$. For the number of gaps broken: against $x^{-1.2}$, the $SSR = 0.003$, and $\chi^2 = 0.016$, whereas against $e^{-.1x}$, the $SSR = 0.019$, and $\chi^2 = 6 \times 10^{11}$. (Note that the chi-squared test statistic is rather large due to the large range of avalanche sizes by gaps broken recorded.)

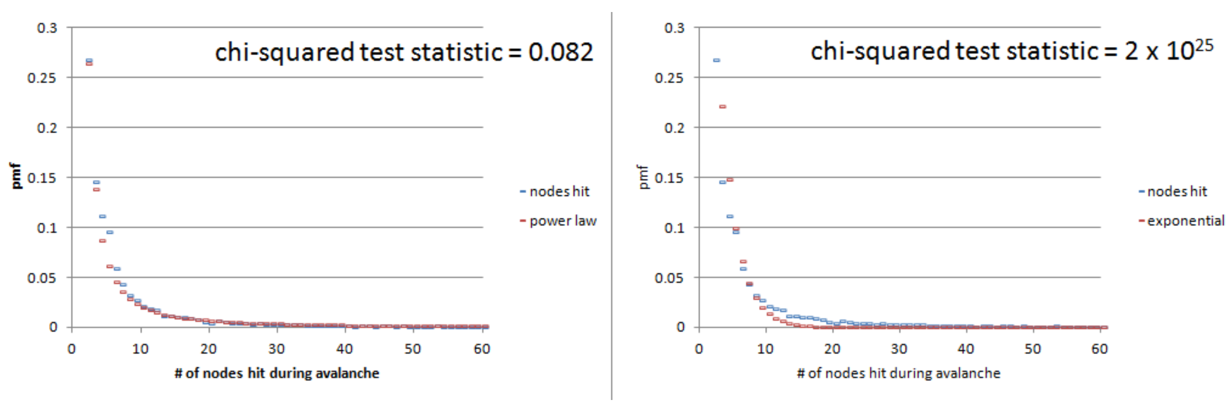
From these simulations, the distributions are very close fits to power laws, which supports the connection to the sandpile model and self-organized criticality. However, since a relatively

Figure 9: 10×10 grid, 1 million random records: number of gaps broken vs $x^{-1.2}$ (left) and vs $e^{-.1x}$ (right)



significant number of the nodes in a 10×10 grid are boundary nodes, which have lesser degree than those inside the grid, we also ran simulations on a 120×120 grid, with results shown in Figure 10. (Larger grids take more time and space, so most simulations were run on smaller grids, with many more records added.) We still see a power law; however, the exponent has decreased slightly (possible reasons for this are discussed in Section 5.1).

Figure 10: 120×120 grid, 15,000 random records: number of distinct nodes hit vs $x^{-1.6}$ (left) and vs $e^{-.4x}$ (right)



5.1 Avalanche Exponents in Two Dimensions

The distributions $D_n(x) \approx x^{-1.3}$ of the number of distinct nodes affected by an avalanche and $D_g(x) \approx x^{-1.2}$ of the number of gaps broken in an avalanche correspond quite closely with the values of numerical estimates made for the exponents of avalanches in the two-dimensional sandpile model for number of distinct sites toppled and number of sites toppled. The sandpile avalanche exponents are expected to be universal, but are still unknown [7]. These estimates tend to fall within $[-2, -1]$, and typically within $[-1.4, -1.2]$ [10, 6, 9, 5]. Priezzhev et al. conjecture that the exact value of the exponent for distinct number of sites toppled is $-\frac{5}{4}$ and that the exact value of the exponent for number of sites toppled is -1.2 [11, 12], which are quite close to -1.3 for distinct number of nodes affected and -1.2 for number of gaps broken in fractional cascading avalanches in two dimensions.

However, a larger grid produced an empirical distribution of $D_n(x) \approx x^{-1.6}$ for avalanches measured by distinct number of nodes affected. That is, a greater proportion of boundary nodes induced larger avalanches more frequently. It may seem intuitive that more nodes with higher degree would give more chances for an avalanche to grow larger, but these results contradict this idea. We also see this kind of behavior in the sandpile model: when the proportion of boundary nodes (to total number of nodes) increases, the exponent increases as well [5, 7]. This suggests that avalanches that hit a boundary are more likely to cycle around the grid again and reach more nodes, overwhelming the effect of a slightly greater degree of freedom.

5.2 Avalanche Exponents in Three Dimensions

Simulations on a $4 \times 4 \times 4$ cubic grid, randomly adding in tens of thousands of records to a fractional cascading structure with maximum gap size $2\Delta(G)$, produced empirical probability distributions $D_n(x) = \mathbf{Pr}[x \text{ distinct nodes are affected by an avalanche}]$ that fit very well with the power law $D_n(x) \approx x^{-1.7}$, and $D_g(x) = \mathbf{Pr}[x \text{ gaps are broken in an avalanche}]$ that fit very well with the power law $D_g(x) \approx x^{-1.3}$. Figures 11 and 12 show these results — $D_n(x)$ and $D_g(x)$, respectively — for a $4 \times 4 \times 4$ grid with 70,000 random records, against both a power law and an exponential distribution.

Figure 11: $4 \times 4 \times 4$ cubic grid, 70,000 random records: number of distinct nodes hit vs $x^{-1.7}$ (left) and vs $e^{-.3x}$ (right)

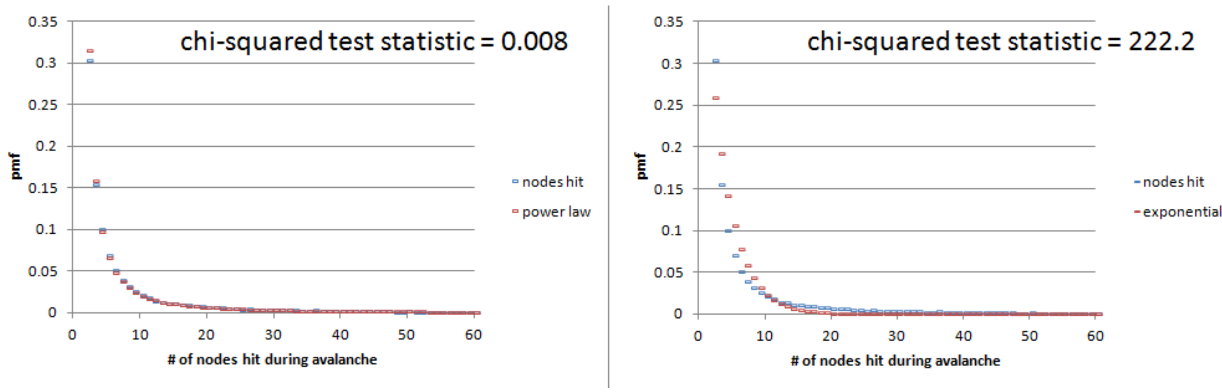
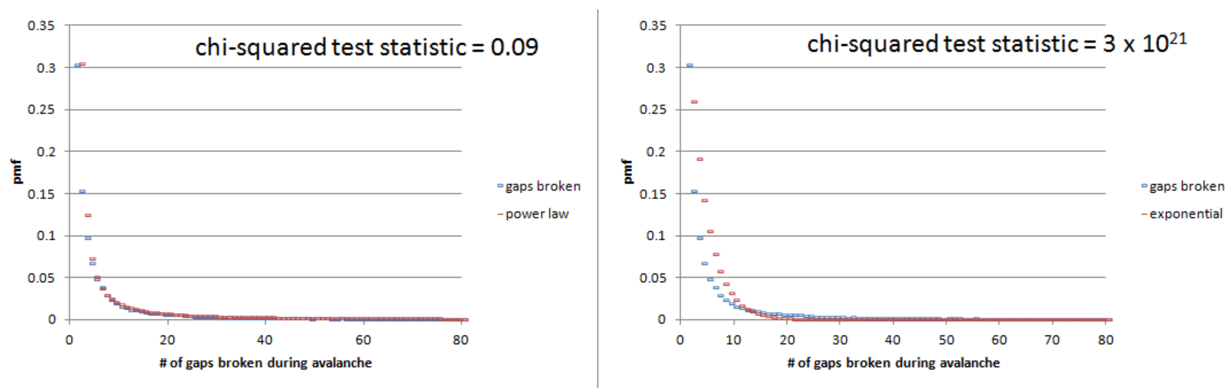


Figure 12: $4 \times 4 \times 4$ cubic grid, 70,000 random records: number of gaps broken vs $x^{-1.3}$ (left) and vs $e^{-.3x}$ (right)



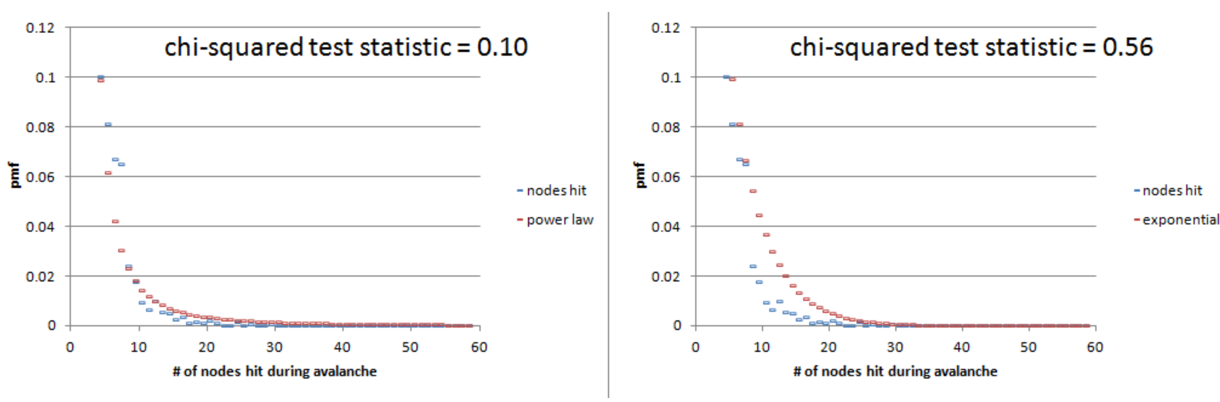
For the number of distinct nodes hit: against $x^{-1.7}$, $SSR = 0.0002$, and $\chi^2 = 0.008$, whereas against $e^{-.3x}$, $SSR = 0.008$, and $\chi^2 = 222.2$. For the number of gaps broken: against $x^{-1.3}$, $SSR = 0.002$, and $\chi^2 = 0.09$, whereas against $e^{-.3x}$, $SSR = 0.009$, and $\chi^2 = 3 \times 10^{21}$. (Again, the chi-squared test statistic is rather large due to the large range of avalanche sizes by gaps broken recorded.)

From these simulations, the distributions are very close fits to power laws. However, the -1.7 exponent for number of distinct nodes hit by an avalanche is somewhat farther from the $-\frac{4}{3}$ exponent for number of distinct sites toppled in the three-dimensional sandpile model, though the assumptions used to compute the sandpile exponents do not all have parallels to fractional cascading — the abelian property of the sandpile model allows for decomposition into waves of toppling and equivalence to the well-known Potts model in classical statistical mechanics, and from the Potts model to other well-studied problems in lattice statistics and graph theory [7, 8]. The -1.3 exponent for number of gaps broken during an avalanche is closer to the $-\frac{4}{3}$ exponent for number of sites toppled in the three-dimensional sandpile model.

We again see that a greater degree of freedom (a greater average and maximum degree in the cubic grid, as opposed to the square grid) does not induce more frequent large avalanches, as the exponent in three dimensions is less than that in two dimensions. This is probably due to the larger maximum gap size (from the higher maximum degree) and boundary nodes of high enough degree that cycling around the grid again is not as significantly more likely than with two-dimensional boundaries.

We do still see that the boundary nodes have influence from simulations on a $25 \times 25 \times 25$ cubic grid with maximum gap size $2\Delta(G)$, as shown in Figure 13. As before, we still see a power law, but with a slightly smaller exponent, -2.1.

Figure 13: $25 \times 25 \times 25$ cubic grid, 3,000 random records: number of distinct nodes hit vs $x^{-2.1}$ (left) and vs $e^{-.2x}$ (right)



5.3 Varying the Gap Size

The previous results were all from simulations carried out with a gap invariant of $2\Delta(G)$, with $\Delta(G)$ denoting the maximum degree of G. However, very small maximum gap sizes allow for the possibility of larger and larger avalanches, which may not always terminate. A very high gap size drastically reduces the probability of large avalanches — doubling the maximum gap size makes it twice as difficult to need to split a gap, and this exponential decay will eventually overcome the power law behavior.

Letting the maximum gap size decrease for a 10×10 grid structure, avalanches too large to handle began to occur under $1.5\Delta(G)$. This is somewhat reminiscent of tuning the probability that a site is open in percolation. Rather than requiring extremely fine tuning of the gap size, though, we find that there is a range of values between $1.5\Delta(G)$ and $2.5\Delta(G)$ with which power law behavior is exhibited. The distribution of the size of avalanches for a fractional cascading structure with gap size above $2.5\Delta(G)$ then tends towards an exponential distribution. Figures 14, 15, and 16 show a progression from power law behavior to exponential, on a 10×10 grid fractional cascading structure. Breaking gaps probabilistically (i.e. if a gap has size greater than the maximum gap size, it is split with probability p) does not affect these thresholds.

Figure 14: 10×10 grid, gap invariant $2.25\Delta(G)$: distribution of size of avalanches against power law and exponential distributions

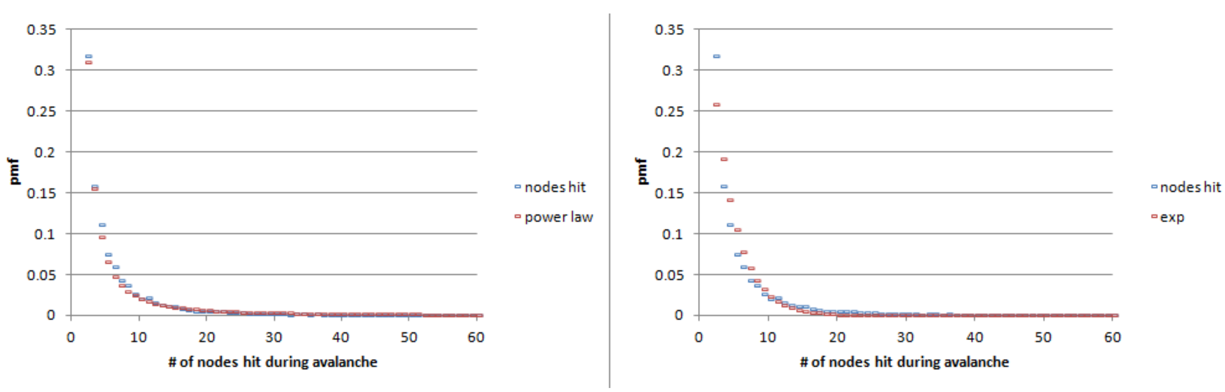


Figure 15: 10×10 grid, gap invariant $3\Delta(G)$: distribution of size of avalanches against power law and exponential distributions

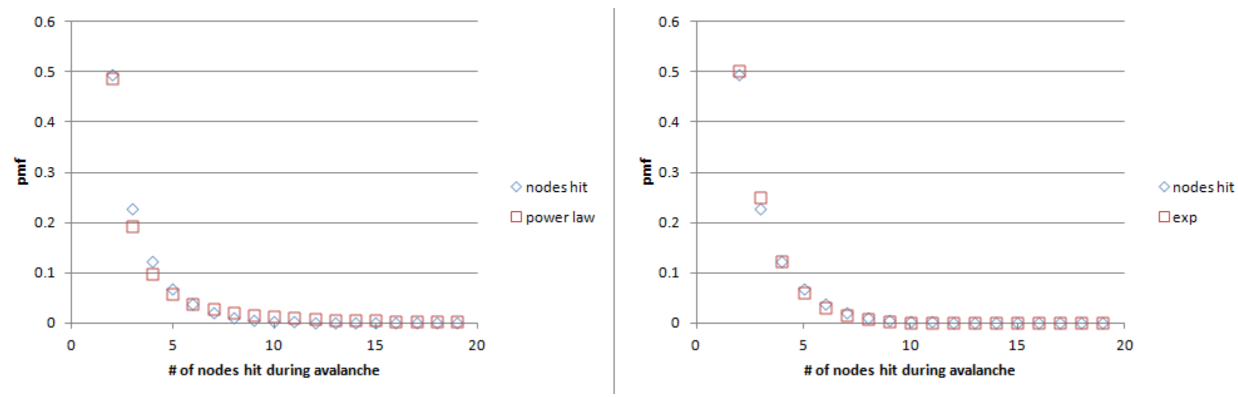
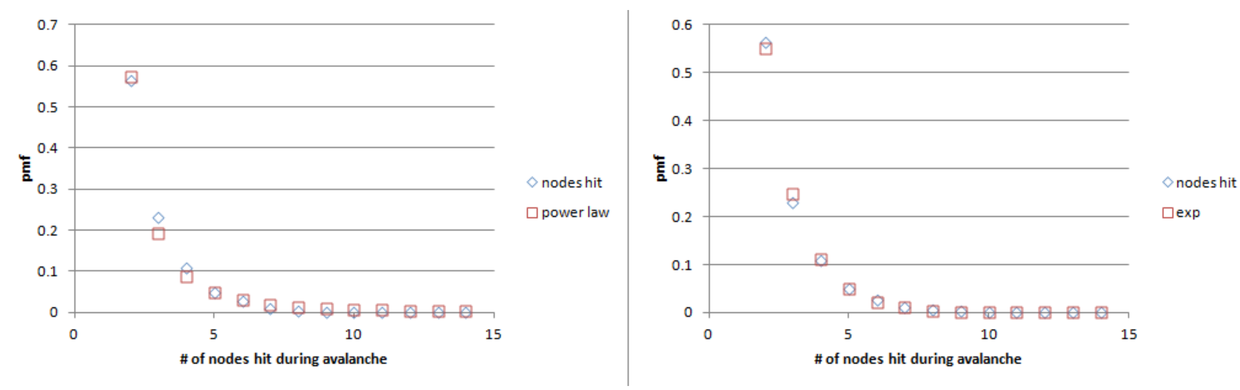


Figure 16: 10×10 grid, gap invariant $3.5\Delta(G)$: distribution of size of avalanches against power law and exponential distributions



6 Conclusions and Future Work

We studied numerically the dynamical properties of fractional cascading in one, two, and three dimensions, and draw strong parallels between the behavior of avalanches in fractional cascading and the avalanches of the Bak-Tang-Wiesenfeld sandpile model. Unlike the sandpile model, however, we prove that fractional cascading does not have abelian properties. Also, where the sandpile model needs no tuning of system parameters, gap size in fractional cascading plays a role akin to temperature in phase transitions — at a very low temperature, a small perturbation will likely cause close to system-wide effects, as with a very low gap size, and at very high temperature, a small perturbation will only have local influence, which we also see with very high gap size — but the gap size does not need to be tuned so precisely as temperature, which needs to be at a very specific value for a system to reach the critical point of a phase transition. In two and three dimensions (i.e. on square and cubic grids) within admissible maximum gap size ranges, we can view fractional cascading as a system with many degrees of freedom with a steady drive, in a non-equilibrium steady state where relaxation (i.e. avalanches) occurs in irregular bursts. This suggests that fractional cascading displays self-organized criticality.

In addition to the strong power law correlations in two and three dimensions, we see some tentative agreement with the power law exponents of the sandpile model, which boundary effects confound. Isolating and confirming these boundary effects should be looked into. Further differences between exponents in the two models are likely due to the more complex nature of state in fractional cascading, taking into account bridges and ordered records. Without an abelian nature to the system, theoretically analyzing the behavior of the system is more difficult, and we cannot as easily make equivalences as with the sandpile model to other well-studied problems, but possible connections should be investigated to a greater extent. Due to equivalences between the sandpile model and several well-studied models, Priezzhev proved that the sandpile model has an upper critical dimension of 4 [13], which is another avenue for study — at least numerically — with fractional cascading. Behavior on other types of lattices (for example, a grid on a cylinder, in which two opposing sides of a grid are linked together, or on a torus, in which the "top" and "bottom" of a grid are linked together as well as the "left" and "right" sides of the grid) could be considered.

Furthermore, additional studying of the macroscopic structure of the avalanches by nodes and edges of the graphs that are affected, and the relation to the size of the avalanche by radius of the affected region may help produce a more concrete conjecture towards the fractional cascading avalanche exponents. To this end, I have coded up a visualization system that marks out avalanches in path graphs, binary trees, and grids, indicating the edges that are part of an avalanche and marking order of processing by color. Preliminary results show relatively compact avalanches, but more analysis should be considered — if they do prove to be compact (or hold some sort of structure), one could consider measuring avalanches by diameter of the affected region (or something characteristic of the structure).

Another aspect to investigate further is any fractal geometry inherent in fractional cascading, as power laws tend to manifest in fractal behavior. Visualizations of the avalanches could help towards identifying fractal behavior. Moreover, where sandpiles simply hold a constant (the height) in the sandpile model, each node of a fractional cascading structure holds a catalogue that is a dynamic structure in and of itself, which opens up the possibility of self-similarity when "zooming in" to the actions on small part of a catalogue. In the one-dimensional case, where the avalanches do not follow power law behavior, we may find this kind of self-similarity, which is not possible in the one-dimensional sandpile model. Such possibilities are quite fascinating, and merit further exploration.

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